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Eight-Vertex Model: Anisotropic Interfacial Tension and Equilibrium Crystal Shape

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The anisotropic interfacial tension of the eight-vertex model is found by a new method, which introduces two inhomogeneous systems. As the width of the system becomes large, a doublet of the largest eigenvalues of the row–row transfer matrix is asymptotically degenerate. The anisotropic interfacial tension is calculated from their finite-size correction terms in this limit. By the use of the anisotropic interfacial tension, the equilibrium crystal shape of the eight-vertex model is derived via Wulff's construction. The equilibrium crystal shape is represented as a simple algebraic curve. We discuss the close relation between the algebraic curve and the form of an elliptic function appearing in the expression of the interfacial tension.

KEY WORDS: Eight-vertex model; row–row transfer matrix; asymptotic degeneracy; interfacial tension; equilibrium crystal shape.

1. INTRODUCTION

In the eight-vertex model an arrow is placed on every edge of a square lattice so that even number of arrows point into and out of each site (or vertex).[1] There are eight such configurations around a vertex (Fig. 1). We represent the arrow configuration by associating an arrow-spin α_i with each edge i; $\alpha_i = +1$ if the corresponding arrow points up or to the right, and $\alpha_i = -1$ otherwise. When arrow-spins around a vertex are ν, α, μ, and β counterclockwise starting from the west bond, a Boltzmann weight $W(\nu, \alpha \mid \beta, \mu)$ is assigned on this vertex, where

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$$\begin{aligned} W(++\mid ++) &= W(--\mid --) = a = \exp(-\varepsilon_1/k_B T) \\ W(+-\mid -+) &= W(-+\mid +-) = b = \exp(-\varepsilon_2/k_B T) \\ W(+-\mid +-) &= W(-+\mid -+) = c = \exp(-\varepsilon_3/k_B T) \\ W(++\mid --) &= W(--\mid ++) = d = \exp(-\varepsilon_4/k_B T) \end{aligned} \tag{1.1a}$$

and

$$W(\nu, \alpha \mid \beta, \mu) = 0, \qquad \nu\alpha\beta\mu = -1 \tag{1.1b}$$

This model was introduced as a generalization of the ice-type (or six-vertex) model. The situation of the ice-type model is as follows. Imagine a square ice where oxygen ions form a square lattice and a hydrogen ion (or a proton) is located near either end of each bond connecting an adjacent pair of oxygen ions. Because of the charge neutrality condition around each site, four protons surrounding each site should satisfy the ice-rule: two of them are close to it and the others are away from it on their respective bonds. By drawing an arrow on every edge, we represent which end of the bond is occupied by a proton. The ice-rule allows six local arrow configurations, which correspond to the vertices 1–6 in the eight-vertex model (Fig. 1). The ice-type model has some unusual features: the antiferroelectric phase transition is infinite order, i.e., the free energy and all its derivatives are finite at the critical temperature; in the ferroelectric ordered state, the ordering is complete even at nonzero temperatures, etc.[2] It is naturally thought that these features come from the ice-rule. To understand the effects of the ice-rule, Sutherland introduced the vertices 7 and 8.[3]

The eight-vertex model can be regarded as an Ising model with two- and four-spin interactions.[1,4] To see this, we associate an Ising spin σ_{ij}

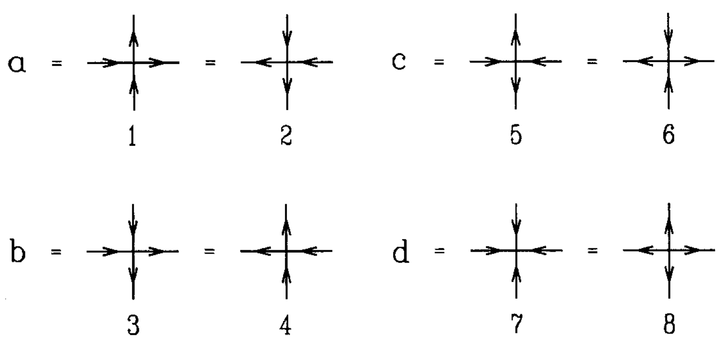

Fig. 1. The eight possible arrow configurations around a vertex, and the corresponding Boltzmann weights.

with each site (i, j) of the dual lattice; the site (i, j) of the dual lattice is connected with the site (i, j) of the original one by shifting in both directions by a half-lattice spacing (Fig. 2). An upward or right arrow corresponds to the cases where the adjacent σ-spins are parallel; otherwise, the adjacent σ-spins are antiparallel. The condition (1.1b) ensures that this correspondence is consistent. To any arrow configuration, there correspond two σ-spin configurations, which are related to each other by the transformation defined by $\sigma_{ij} \to -\sigma_{ij}$ for all i, j. The Hamiltonian of the Ising model is given by

$$E = -\sum_{ij} (J_1 \sigma_{i,j+1} \sigma_{i+1,j} + J_2 \sigma_{ij} \sigma_{i+1,j+1} + J_3 \sigma_{ij} \sigma_{i+1,j} \sigma_{i+1,j+1} \sigma_{i,j+1}) \tag{1.2}$$

where next-nearest-neighbor spins are coupled by J_1 and J_2, depending on the direction of the diagonal; J_3 couples four spins on a unit square. The four Boltzmann weights in (1.1a) are related to J_1, J_2, and J_3 as follows:

$$\exp\left[\frac{4J_1}{k_B T}\right] = \frac{ad}{bc}, \qquad \exp\left[\frac{4J_2}{k_B T}\right] = \frac{ac}{bd}, \qquad \exp\left[\frac{4J_3}{k_B T}\right] = \frac{ab}{cd} \tag{1.3}$$

When a, b, c, and d satisfy the relation

$$ab = cd \tag{1.4}$$

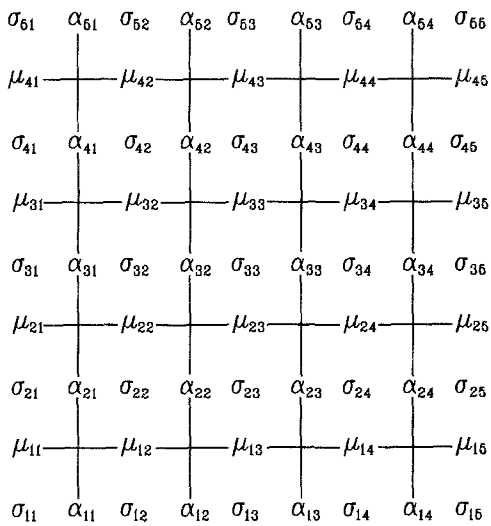

Fig. 2. The arrow-spins α and β are associated with the edges of the square lattice; the Ising spins σ are associated with the sites of the dual lattice.

this Ising model factors into two independent nearest-neighbor Ising models.

Recently, equilibrium crystal shapes (ECSs) have attracted much attention. The first exact analysis of ECSs was done for the square-lattice nearest-neighbor Ising model by Rottman and Wortis[5] and Avron *et al.*[6] The traditional method to find the ECS is the so-called Wulff construction, where we need information about the interfacial tension with its full anisotropy.[7] For the square-lattice nearest-neighbor Ising model, it has been proved that the anisotropic interfacial tension is simply related to the anisotropic correlation length.[8] Furthermore, the anisotropic interfacial tension has been calculated by the Pfaffian method.[9] By the use of these results, the ECS has been derived via Wulff's construction. Zia and Avron showed that the ECS of this model can be represented as a simple algebraic curve.[10] For example, when the interactions are isotropic, the ECS is given by

$$\cosh(\Lambda X/k_B T) + \cosh(\Lambda Y/k_B T) = C_I \tag{1.5a}$$

$$C_I = \cosh(2J/k_B T)/\tanh(2\,|J|/k_B T) \tag{1.5b}$$

where (X, Y) is the position vector of a point on the ECS; Λ is a scale factor; and J is the interaction constant. Zia similarly found that the ECSs of the triangular and honeycomb lattice nearest-neighbor Ising models are also written as an algebraic curve like (1.5).[11] For any Ising model on the planar lattice without bond crossings, Holzer[12] and Akutsu and Akutsu[13] pointed out that the interface can be represented as a free random walk defined on the dual lattice. The free random walk is derived by the use of the Feynman–Vdovichenko method[14] and characterized by the algebraic curve of the ECS.

van Beijeren[15] and Jayaprakash *et al.*[16] regarded the six-vertex model as a solid-on-solid model on a body-centered-cubic lattice (BCSOS model), and discussed the roughening transition of a three-dimensional crystal. They utilized the fact that the six-vertex model can be solved in external field.[17] Akutsu and Akutsu reexamined the facet shape of the BCSOS model (or equivalently the ECS of the six-vertex model).[13] They found that the facet shape can be written in the same form (1.5a), with C_I replaced by

$$C_{BC} = k^{1/2}(x) + k^{-1/2}(x) \tag{1.6a}$$

where $k(x)$ is defined as

$$k(x) = 4x \prod_{n=1}^{\infty} \left[\frac{1 + x^{4n}}{1 + x^{4n-2}} \right]^4 \tag{1.6b}$$

and x is given by (3.5). Since the BCSOS model cannot be solved by the Feynman–Vdovichenko method, the free-random-walk representation is not directly applicable to the step of this model. Discussing long-distance behavior, Akutsu and Akutsu suggested that the step of the BCSOS model has the same free-random-walk character as the interface of the square-lattice nearest-neighbor Ising model.

In connection with these free-random-walk problems, it is desirable to analyze the ECS of the eight-vertex model, which contains the square-lattice nearest-neighbor Ising model and the six-vertex (or BCSOS) model as special limits. For the eight-vertex model, Baxter obtained the interfacial tension along a special direction by the usual transfer matrix method.[1,18] The usual transfer matrix method, however, is not applicable to the calculation of the anisotropy. In previous work, to find the anisotropic correlation length and the anisotropic interfacial tension of the hard-hexagon model, we developed a new method which introduces the shift operator into the usual transfer matrix method.[19,20] In the present paper we calculate the anisotropic interfacial tension of the eight-vertex model by this shift operator method. Then, the ECS of the eight-vertex model is derived via Wulff's construction.

This paper is organized as follows. In Section 2 we explain the shift operator method of calculating the anisotropic interfacial tension of the eight-vertex model. Two inhomogeneous systems are defined; each inhomogeneous system consists of two regions; one of the two regions works as the (column–column) shift operator. These inhomogeneous systems are analyzed by Baxter's commuting transfer matrices argument.[1,21] We introduce a one-parameter family of commuting transfer matrices. Then an equation for the eigenvalues of the transfer matrix is derived. In Section 3, we outline the calculation of solving the equation and give the explicit forms of the eigenvalues; details of the calculation is shown in Appendix B. It is found that a doublet of the largest eigenvalues is asymptotically degenerate when the width of the system becomes large. The anisotropic interfacial tension is obtained from the finite-size correction terms of the doublet of the largest eigenvalues in this limit. In Section 4, using the calculated interfacial tension, we find the ECS via Wulff's construction. We discuss the relation between the ECS and the elliptic solution of the interfacial tension given in Section 3. Section 5 is devoted to a summary and discussion. Appendix A lists definitions of elliptic functions and relations among them.

2. ANISOTROPIC INTERFACIAL TENSION 1

In this section we explain a method to calculate the anisotropic interfacial tension of the eight-vertex model. We introduce two inhomogeneous systems.[20] Then we show how these inhomogeneous systems are investigated by Baxter's commuting transfer matrices argument.[1,21] In the following analysis, it is convenient to parametrize the Boltzmann weights (1.1a) by four variables ρ, k, λ, and u:

$$\begin{aligned} a &= -i\rho\Theta(i\lambda)\, H[i(\lambda-u)/2]\, \Theta[i(\lambda+u)/2] \\ b &= -i\rho\Theta(i\lambda)\, \Theta[i(\lambda-u)/2]\, H[i(\lambda+u)/2] \\ c &= -i\rho H(i\lambda)\, \Theta[i(\lambda-u)/2]\, \Theta[i(\lambda+u)/2] \\ d &= i\rho H(i\lambda)\, H[i(\lambda-u)/2]\, H[i(\lambda+u)/2] \end{aligned} \tag{2.1}$$

where ρ is a normalization factor of the partition function and λ, u appear as arguments of the elliptic functions. The elliptic modulus k is suppressed here. We denote the quarter-periods by I and I'. For definitions of the elliptic functions, see Appendix A.

Consider the eight-vertex model on a square lattice with $(1+\eta)M$ columns and N rows $[(1+\eta)M, N$ even$]$. It is assumed that ρ and u can vary from column to column.[20] The values of ρ and u on the jth column are denoted by ρ_j and u_j, respectively. We define the anisotropic interfacial tension, using two inhomogeneous systems:

$$\begin{aligned} &u_1 = u_2 = \cdots = u_M = u_0, \qquad u_{M+1} = u_{M+2} \\ &\quad = \cdots = u_{(1+\eta M} = \mp\lambda \\ &\rho_1 = \rho_2 = \cdots = \rho_M = \rho_0, \qquad \rho_{M+1} = \rho_{M+2} = \cdots = \rho_{(1+\eta)M} \\ &\quad = i/\Theta(0)\, \Theta(i\lambda)\, H(i\lambda) \end{aligned}$$

The system with the upper (lower) sign is called (A) [(B)]. Because of various symmetry properties, we restrict ourselves to the parameter regime

$$0<k<1, \qquad 0<\lambda<I', \qquad -\lambda<u_0<\lambda, \qquad \rho_0>0 \tag{2.2}$$

without loss of generality.[1,22] The region where $u_j = u_0$ for $1<j<M$ is in an antiferroelectric ordered state dominated by the vertices 5 and 6.

We impose on (A) and (B) periodic boundary conditions along the horizontal direction and antiperiodic boundary conditions along the vertical direction. Then, the antiperiodic boundary conditions force an interface into the region $u_j = u_0$. In the region $u_j = -\lambda$ (or λ) for $M+1<j<(1+\eta)M$, the arrow configuration around the site (j, k) is identical with that of the site $(j+1, k+1)$ [or $(j+1, k-1)$]. Therefore, the region $u_j = -\lambda$ (or λ) slopes the interface by shifting the endpoint on the Mth column from the endpoint on the first column downward (or upward) by $M\eta$ lattice spacings (Fig. 3). We represent the average slope of

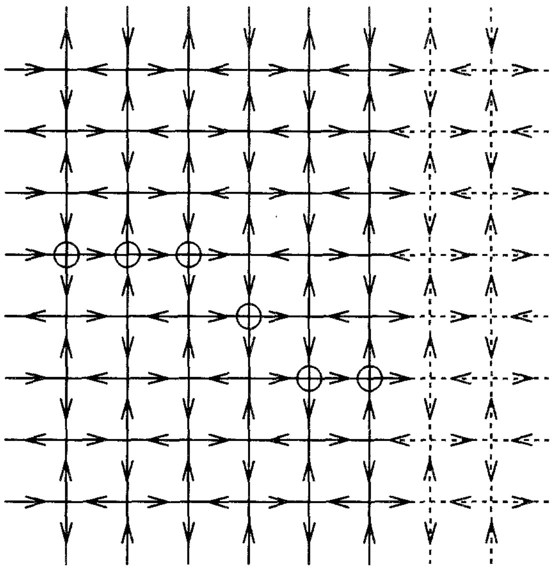

Fig. 3. Typical configuration of the inhomogeneous system (A) in the zero-temperature limit. The region $u_j = u_0$ is represented by solid lines, and the region $u_j = -\lambda$ by broken lines. In the region $u_j = u_0$ two antiferroelectric ordered phases dominated by the vertices 5 and 6 coexist. Across the region $u_j = u_0$, there is an interface. The interface consists of the vertices 1 and 3, which are shown by open circles. Because of the region $u_j = -\lambda$, the interface is sloped.

the interface by $\theta_\perp$, which is the angle between the horizontal axis and the normal vector of the line connecting the two endpoints of the interface drawn from the lower phase toward the upper one. The parameter η is related to $\theta_\perp$ by

$$\text{(A)}\quad \eta = 1/\tan\theta_\perp, \qquad 0 < \theta_\perp < \pi/2 \tag{2.3a}$$

$$\text{(B)}\quad \eta = -1/\tan\theta_\perp, \qquad \pi/2 < \theta_\perp < \pi \tag{2.3b}$$

Let $Z^{(1)}_{MN}(\theta_\perp)$ $(0 < \theta_\perp < \pi)$ be the partition function of (A) and (B) with the boundary conditions. If periodic boundary conditions are imposed along both directions, we denote corresponding partition function by $Z^{(0)}_{MN}$. The anisotropic interfacial tension $\sigma(\theta_\perp)$ is defined as

$$\sigma(\theta_\perp) = -k_B T \sin\theta_\perp \lim_{M,N\to\infty} M^{-1} \ln[Z^{(1)}_{MN}(\theta_\perp)/Z^{(0)}_{MN}], \qquad 0 < \theta_\perp < \pi \tag{2.4}$$

where the limit is taken with $\theta_\perp$ (or η) fixed to be constant.

When Baxter solved the eight-vertex model, he found a one-parameter family of commuting transfer matrices.[1,21] Then he used an equation for

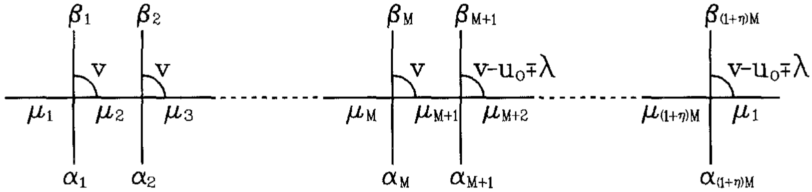

Fig. 4. A row of the generalized system (A) [or (B)].

the transfer matrix to determine the explicit forms of their eigenvalues. He suggested that the commuting transfer matrices argument is applicable to the inhomogeneous systems where u and ρ can vary from column to column. The partition functions of (A) and (B) in (2.4) are calculated by Baxter's commuting transfer matrices argument as follows. First, to introduce a one-parameter family of commuting transfer matrices, the systems (A) and (B) are generalized: we set the u_j to be

$$u_1 = u_2 = \cdots = u_M = v$$

$$u_{M+1} = u_{M+2} = \cdots = u_{(1+\eta)M} = v - u_0 \mp \lambda$$

Hereafter, unless otherwise mentioned, we regard v as a complex variable and k, λ, u_0, and ρ_0 as constants. The upper (lower) sign corresponds to the generalized system (A) [(B)]. (Throughout this section, we use this convention.) Let $\alpha = \{\alpha_1, \alpha_2, ..., \alpha_{(1+\eta)M}\}$ and $\beta = \{\beta_1, \beta_2, ..., \beta_{(1+\eta)M}\}$ be the arrow-spins on two successive rows of vertical edges; let $\mu = \{\mu_1, \mu_2, ..., \mu_{(1+\eta)M}\}$ be the arrow-spins on the row intervening between α and β (Fig. 4). The row–row transfer matrix is a $2^{(1+\eta)M}$ by $2^{(1+\eta)M}$ matrix with the elements

$$[\mathbf{V}(v)]_{\alpha,\beta} = \sum_{\mu} \prod_{j=1}^{M} W(\mu_j, \alpha_j \mid \beta_j, \mu_{j+1} \mid v) \times \prod_{k=M+1}^{(1+\eta)M} W(\mu_k, \alpha_k \mid \beta_k, \mu_{k+1} \mid v - u_0 \mp \lambda) \tag{2.5}$$

where W's are given by (1.1) with (2.1); periodic boundary conditions along the horizontal direction are assumed: $\mu_{(1+\eta)M+1} = \mu_1$. For all complex numbers v and v', $\mathbf{V}(v)$ and $\mathbf{V}(v')$ commute with each other, being simultaneously diagonalized. The eigenvalues of $\mathbf{V}(v)$ are denoted by $V(v)$.

Second, an equation for $V(v)$ is derived. We define a set of $2^{(1+\eta)M}$-dimensional vectors $\mathbf{y}(v)$. Each vector $\mathbf{y}(v)$ is of the form

$$\mathbf{y}(v) = \mathbf{g}_1(v) \otimes \mathbf{g}_2(v) \otimes \cdots \otimes \mathbf{g}_M(v) \otimes \mathbf{g}_{M+1}(v - u_0 \mp \lambda) \otimes \cdots \otimes \mathbf{g}_{(1+\eta)M}(v - u_0 \mp \lambda) \tag{2.6}$$

where

$$\mathbf{g}_j(v)=\begin{pmatrix} g_j(+\mid v) \\ g_j(-\mid v)\end{pmatrix}=\begin{pmatrix} \Theta[is_j+i(v+\lambda)\,\zeta_j/2] \\ (-)^{j+1}\, iH[is_j+i(v+\lambda)\,\zeta_j/2]\end{pmatrix} \tag{2.7}$$

Each ζ_j has a value $+1$ or -1 and the ζ_j satisfy the condition

$$\zeta_1+\zeta_2+\cdots+\zeta_{(1+\eta)M}=0 \tag{2.8}$$

The variables s_j are given by

$$s_j=\begin{cases}\bar{s}, & j=1 \\ \bar{s}+\lambda(\zeta_1+\zeta_2+\cdots+\zeta_{j-1}), & j=2,3,\ldots,(1+\eta)\,M\end{cases} \tag{2.9}$$

with an arbitrary constant $\bar{s}$. It can be shown that

$$\mathbf{V}(v)\,\mathbf{y}(v)=\phi_1(v)\,\mathbf{y}(v+2\lambda')+\phi_2(v)\,\mathbf{y}(v-2\lambda') \tag{2.10}$$

where $\lambda'=\lambda-2iI$ and

$$\phi_1(v)=\{\rho_1 h[(\lambda-v)/2]\}^M\,\{\rho_{M+1}h[(\lambda-v+u_0\pm\lambda)/2]\}^{M\eta} \tag{2.11a}$$

$$\phi_2(v)=\{\rho_1 h[(\lambda+v)/2]\}^M\,\{\rho_{M+1}h[(\lambda+v-u_0\mp\lambda)/2]\}^{M\eta} \tag{2.11b}$$

$$h(v)=-i\Theta(0)\,\Theta(iv)\,H(iv) \tag{2.11c}$$

There exist many $2^{(1+\eta)M}$-dimensional vectors $\mathbf{y}(v)$, corresponding to different choices of $\bar{s}$ and the ζ_j. Using a complete set of vectors $\mathbf{y}(v)$, we can construct a nonsingular matrix $\mathbf{Q}(v)$ which satisfies the matrix equation

$$\mathbf{V}(v)\,\mathbf{Q}(v)=\phi_1(v)\,\mathbf{Q}(v+2\lambda')+\phi_2(v)\,\mathbf{Q}(v-2\lambda') \tag{2.12}$$

Since $\mathbf{Q}(v)$ commutes with $\mathbf{Q}(v')$ and $\mathbf{V}(v'')$ for all values of v, v', and v'', we get the equation for $V(v)$

$$V(v)\,q(v)=\phi_1(v)\,q(v+2\lambda')+\phi_2(v)\,q(v-2\lambda') \tag{2.13}$$

where $q(v)$ is the eigenvalue of $\mathbf{Q}(v)$ corresponding to $V(v)$.

Two matrices **S** and **R** are introduced; **S** is the diagonal matrix which has entries $+1$ (or -1) for arrow configurations of an even (or odd) number of down arrows; **R** is the matrix which has the effect of reversing all arrows; we shall use **R** to impose the antiperiodic boundary conditions along the vertical direction. The matrices **S**, **R**, $\mathbf{Q}(v)$, and $\mathbf{V}(v')$ commute with each other for all values of v and v'. The eigenvalue of **S** (or **R**) corresponding to $V(v)$ and $q(v)$ is denoted by s (or r); r takes a value of $+1$ or -1. Detailed investigation shows that $q(v)$ must be of the form

$$q(v) = \exp(\tau v) \prod_{j=1}^{m} h\left(\frac{v - v_j}{2}\right) \tag{2.14}$$

where $m = (1 + \eta) M/2$. The zeros v_j and a constant τ are determined by the condition that the rhs of (2.13) vanishes,

$$\left\{\frac{h[(\lambda - v_j)/2]}{h[(\lambda + v_j)/2]}\right\}^M \left\{\frac{h[(\lambda - v_j + u_0 \pm \lambda)/2]}{h[(\lambda + v_j - u_0 \mp \lambda)/2]}\right\}^{M\eta}$$
$$= -\exp(-4\tau\lambda') \prod_{k=1}^{m} \frac{h[(v_j - v_k - 2\lambda)/2]}{h[(v_j - v_k + 2\lambda)/2]} \tag{2.15}$$

for $j = 1, 2, \ldots, m$, and the sum rules

$$v_1 + v_2 + \cdots + v_m - \frac{M}{2}\eta(u_0 \pm \lambda)$$
$$= \frac{1}{2}(s - 1 + 2m) I' + i(rs - 1 + 2m) I + 2p'I' + 4piI \tag{2.16a}$$

$$\tau = \pi(s - 1 + 2m + 4p')/8I \tag{2.16b}$$

where p, p' are integers. The sum rules (2.16) come from the periodicity relations

$$q(v + 4iI) = sq(v) \tag{2.17a}$$

$$q(v + 2I') = rsq^{-m/2} \exp\{[M(1 + \eta) v - M\eta\lambda] \pi/4I\} q(v) \tag{2.17b}$$

The eigenvalues $V(v)$ can be calculated by solving (2.15) with (2.16), and then by using (2.14) in (2.13) with the solutions v_j and τ. There are many eigenvalues, corresponding to the different solutions of (2.15). After the eigenvalues $V(v)$ are determined, we can get the information needed to obtain the anisotropic interfacial tension by letting $v = u_0$. The two partition functions in (2.4) are represented as

$$Z_{MN}^{(n)} = \mathrm{Tr}[\mathbf{V}^N(u_0)\, \mathbf{R}^n] = \sum_j V_j^N(u_0)\, r_j^n \tag{2.18}$$

where $V_j(u_0)$ and r_j are the jth eigenvalues of $\mathbf{V}(u_0)$ and $\mathbf{R}$, respectively. Note that $\mathbf{R}$ is inserted to impose two different of boundary conditions along the vertical direction: periodic boundary conditions for $n \equiv 0 \pmod 2$ and antiperiodic boundary conditions for $n \equiv 1 \pmod 2$.

3. ANISOTROPIC INTERFACIAL TENSION 2

Following the program given in Section 2, we find the anisotropic interfacial tension of the eight-vertex model. The calculation in this section is an extension of Baxter.[1,18] Since the calculational method is almost the same for any value of u_0, it is sufficient to consider the case $u_0=0$. In the case $u_0=0$, the generalized system (A) contains both original systems (A) and (B): the original system (A) corresponds to the $v\to 0$ limit; if we redefine the ρ_j as

$$\rho_1=\rho_2=\cdots=\rho_M=i/\Theta(0)\,\Theta(i\lambda)\,H(i\lambda)$$

$$\rho_{M+1}=\rho_{M+2}=\cdots=\rho_{(1+\eta)M}=\rho_0$$

the generalized system (A) reduces to the original system (B) as $v\to\lambda$. For the system (B) given by the $v\to\lambda$ limit, (2.3b) should be replaced by

$$\text{(B)}\quad \eta=-\tan\theta_\perp,\qquad \pi/2<\theta_\perp<\pi \tag{2.3b$'$}$$

We investigate the commuting transfer matrices argument for the generalized system (A) only. Equations (2.13)–(2.16) are used with $u_0=0$ and the upper sign. For convenience, we define $P(v)$ by

$$P(v)=\phi_1(v)\,q(v+2\lambda')/\phi_2(v)\,q(v-2\lambda') \tag{3.1}$$

and rewrite (2.13) as

$$V(v)\,q(v)=\phi_2(v)\,q(v-2\lambda')[1+P(v)] \tag{3.2a}$$

$$=\phi_1(v)\,q(v+2\lambda')[1+1/P(v)] \tag{3.2b}$$

Analysis is restricted to the parameter regime

$$0<k<1,\qquad 0<\lambda<I',\qquad 0<\mathrm{Re}(v)<\lambda,\qquad \rho_0>0 \tag{3.3}$$

In the regime (3.3) two antiferroelectric ordered states dominated by the vertices 5 and 6 are degenerate. It is expected that a doublet of the largest eigenvalues of $\mathbf{V}(v)$ is asymptotically degenerate as $M\to\infty$ (with η fixed to be constant). These eigenvalues are found in a zero-temperature analysis. Then we return to nonzero temperatures and determine their asymptotic forms as $M\to\infty$. Using the asymptotic forms, we estimate the two partition functions $Z^{(n)}_{MN}$ in (2.4) for large M, N. The anisotropic interfacial tension is given by the finite-size correction terms of the doublet of the largest eigenvalues in the $M\to\infty$ limit.

In the parametrization (2.1) the zero-temperature limit corresponds to the $k\to 0$ and I', λ, $v\to\infty$ limit, with the ratios λ/I', v/I' being order of unity. It is supposed that $0 < \mathrm{Re}(v_j) < \lambda$ for all j. Then, the zero-temperature analysis of (2.15) and (2.16) shows that

$$\tau = 0, \qquad s = (-1)^m, \qquad \prod_{j=1}^{m} z_j = r x^{M\eta/2} \tag{3.4}$$

where

$$z_j = \exp(-\pi v_j/2I), \qquad x = \exp(-\pi\lambda/2I) \tag{3.5}$$

Because of the sum rules (2.16), τ and $\prod_{j=1}^{m} z_j$ take discrete values. The eigenvalue s takes a value of $+1$ or -1. We expect that τ, s, and $\prod_{j=1}^{m} z_j$ keep their zero-temperature values (3.4) throughout the regime (3.3) without discontinuous changes. According as $r = +1$ or -1, the v_j and $P(v)$ behave in the zero-temperature limit as

$$v_j \sim \frac{2I}{m} i \left[2j - m - \frac{(r+1)}{2} \right] + \frac{\eta}{1+\eta} \lambda \tag{3.6}$$

$$P_r(v) \sim r(-1)^m z^m x^{-M\eta/2}$$

with

$$-\min\left\{0, 2 - \frac{2+3\eta}{1+\eta}\frac{\lambda}{I'}\right\} < \mathrm{Re}\left(\frac{v}{I'}\right) < \min\left\{\frac{\lambda}{I'}, 2 - \frac{2+\eta}{1+\eta}\frac{\lambda}{I'}\right\} \tag{3.7a}$$

$$\sim q^m x^{-2m}, \qquad 2 - \frac{2+\eta}{1+\eta}\frac{\lambda}{I'} < \mathrm{Re}\left(\frac{v}{I'}\right) < \frac{\lambda}{I'} \tag{3.7b}$$

$$\sim q^{-m} x^{2m}, \qquad 0 < \mathrm{Re}\left(\frac{v}{I'}\right) < \frac{2+3\eta}{1+\eta}\frac{\lambda}{I'} - 2 \tag{3.7c}$$

where

$$z = \exp(-\pi v/2I) \tag{3.8}$$

The regions of applicability of the three conditions (3.7a)–(3.7c) are shown in Fig. 5. Using (3.6) and (3.7) in (3.2), we find two eigenvalues

$$V_r(v) \sim r\rho_1^M \rho_{M+1}^{M\eta} q^{m/2} x^{-2m} \sim r \prod_{j=1}^{(1+\eta)M} c_j, \qquad r = \pm 1 \tag{3.9}$$

The Boltzmann weights a_j, b_j, c_j, and d_j are given by (2.1), with $\rho = \rho_j$ and $u = u_j$. Note that in the zero-temperature limit $c_j \gg a_j, b_j, d_j$ for all j. We identify the two eigenvalues $V_r(v)$ as the doublet of the largest eigenvalues.

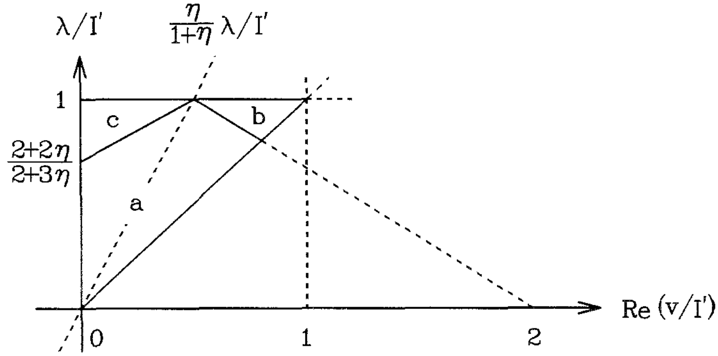

Fig. 5. The regions of the applicability of the three conditions (3.7a)–(3.7c).

Here, we return to nonzero temperatures and calculate the asymptotic behavior of $P_r(v)$ and $V_r(v)$ as $M \to \infty$. The zero-temperature results (3.7) and (3.9) satisfy the following conditions:

(i) For large M a contour C defined by $|P_r(v)| = 1$ is found in the region $0 < \mathrm{Re}(v) < \lambda$; the v_j lie on the contour C.

(ii) There exists a real, positive number δ such that $P_r(v)$ is exponentially larger than 1 as $M \to \infty$ if v is between the contour C and the line $\mathrm{Re}(v) = -\delta$; $P_r(v)$ is exponentially smaller than 1 if v is between the contour C and the line $\mathrm{Re}(v) = \lambda + \delta$.

(iii) $V_r(v)$ is analytic and nonzero (ANZ) for $0 < \mathrm{Re}(v) < \lambda$.

We assume that (i)–(iii) hold true for sufficiently low temperatures: $0 < k < \varepsilon$. Then, after some calculations, it is found that for $0 < k < \varepsilon$ and large M

$$P_r(v) \sim r p^M(v)\, p^{M\eta}(v - \lambda), \qquad v \in \text{the region } a \tag{3.10a}$$

$$\sim 1, \qquad v \in \text{the region } b \tag{3.10b}$$

$$\sim p^M(v)\, p^M(v - 2I')\, p^{M\eta}(v - \lambda) \times p^{M\eta}(v - \lambda - 2I'), \qquad v \in \text{the region } c \tag{3.10c}$$

where

$$p(v) = (-z)^{1/2} f(xz^{-1}, x^4)/f(xz, x^4) \tag{3.11}$$

and $f(z, x^4)$ is given by (A.6a) in Appendix A. In the complex v plane, the three regions a, b, and c are defined as follows: for a given point v, choose a point v_C on C such that $\mathrm{Im}(v) = \mathrm{Im}(v_C)$; v is in the region a if

$|v-v_C|<\min\{2\lambda, 2I'-2\lambda\}$, in the region b if $2\lambda<v-v_C<2I'-2\lambda$, in the region c if $2I'-2\lambda<v-v_C<2\lambda$ (Fig. 6). Because of the periodicity relations $P_r(v+2I')=P_r(v+4iI)=P_r(v)$, Eqs. (3.10) determine $P_r(v)$ for all v. For $0<k<\varepsilon$, we also find that as $M\to\infty$,

$$V_r(v)\sim r\kappa(v), \qquad 0<\mathrm{Re}(v)<\lambda \tag{3.12}$$

where

$$\kappa(v)=\left(\frac{\gamma\rho_1}{x}\right)^M \prod_{n=0}^{\infty}\frac{A[v+(4n+3)\lambda]\,A[v+2I'+(4n-1)\lambda]}{A[v+(4n+5)\lambda]\,A[v+2I'+(4n+1)\lambda]}$$
$$\times\prod_{n=0}^{\infty}\frac{A[(4n+3)\lambda-v]\,A[(4n-1)\lambda-v+2I']}{A[(4n+5)\lambda-v]\,A[(4n+1)\lambda-v+2I']}$$
$$\times\left(\frac{\gamma\rho_{M+1}}{x}\right)^{M\eta}\prod_{n=0}^{\infty}\frac{A^\eta[v+(4n+2)\lambda]\,A^\eta[v+2I'+(4n-2)\lambda]}{A^\eta[v+(4n+4)\lambda]\,A^\eta[v+2I'+4n\lambda]}$$
$$\times\prod_{n=0}^{\infty}\frac{A^\eta[(4n+4)\lambda-v]\,A^\eta[4n\lambda-v+2I']}{A^\eta[(4n+6)\lambda-v]\,A^\eta[(4n+2)\lambda-v+2I']} \tag{3.13}$$

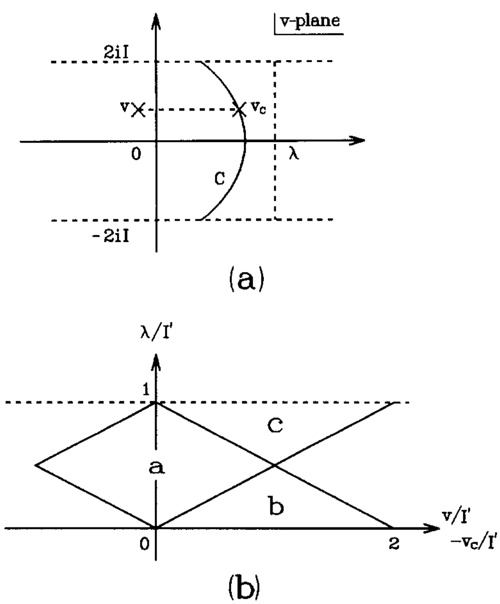

Fig. 6. The three regions a, b, and c in (3.10). (a) For a given point v, choose a point v_C on the contour C so that $\mathrm{Im}(v)=\mathrm{Im}(v_C)$. (b) Then, using $v-v_C$, we define three regions a, b, and c.

with

$$\gamma = q^{1/4}\Theta(0) \prod_{n=1}^{\infty} (1-q^{2n})^2 \tag{3.14}$$

$$A(v) = \prod_{n=0}^{\infty} [1 - q^n \exp(-\pi v/2I)]^M \tag{3.15}$$

In the $k \to 0$ limit, (3.10) reproduces the zero-temperature result (3.7); (3.12) reproduces (3.9). Furthermore, the solutions (3.10) and (3.12) satisfy the three conditions (i)–(iii) for $0 < k < 1$. These facts show that (3.10) and (3.12) give the correct leading behavior of $V_r(v)$ and $P_r(v)$ as $M \to \infty$, respectively, throughout the regime (3.3). [Detailed derivations of (3.10) and (3.12) are shown in Appendix B.]

In (3.12) the leading terms of $V_r(v)$ as $M \to \infty$ are equal in magnitude and opposite in sign. From the calculation of the finite-size correction terms, it is found that they are asymptotically degenerate as $M \to \infty$.[18] For $0 < \lambda < I'/2$, the finite-size correction terms are easily obtained. If M is finite, we get the integral equation

$$\begin{aligned} \ln\left[\frac{rV_r(v)}{\kappa(v)}\right] &= \frac{1}{8iI}\int_{\lambda-2iI}^{\lambda+2iI} dv' \ln[1+P_r(v')]\, D(v-v') \\ &\quad - \frac{1}{8iI}\int_{-2iI}^{+2iI} dv' \ln\left[1+\frac{1}{P_r(v')}\right] D(v-v'), \qquad 0 < \operatorname{Re}(v) < \lambda \end{aligned} \tag{3.16}$$

where

$$D(v) = 1 + 2\sum_{n=0}^{\infty} (-1)^n \left\{\frac{x^{2n}z^{-1}}{1-x^{2n}z^{-1}} + \frac{x^{2n+2}z}{1-x^{2n+2}z}\right\} \tag{3.17}$$

(Appendix B). Subtracting (3.16) with $r = -1$ from that with $r = +1$ gives

$$\begin{aligned} \ln\left[\frac{-V_+(v)}{V_-(v)}\right] &= \frac{1}{8iI}\int_{\lambda-2iI}^{\lambda+2iI} dv' \ln\left[\frac{1+P_+(v')}{1+P_-(v')}\right] D(v-v') \\ &\quad - \frac{1}{8iI}\int_{-2iI}^{+2iI} dv' \ln\left[\frac{1+1/P_+(v')}{1+1/P_-(v')}\right] D(v-v'), \qquad 0 < \operatorname{Re}(v) < \lambda \end{aligned} \tag{3.18}$$

For large M, using (3.10a), we estimate the logarithms in the integrands of (3.18) as

$$\ln\left[\frac{1+P_+(v')}{1+P_-(v')}\right] \sim P_+(v') - P_-(v') \sim 2p^M(v')\, p^{M\eta}\,(v'-\lambda) \tag{3.19a}$$

$$\ln\left[\frac{1+1/P_+(v')}{1+1/P_-(v')}\right] \sim \frac{1}{P_+(v')} - \frac{1}{P_-(v')} \sim \frac{2}{p^M(v')\, p^{M\eta}\,(v'-\lambda)} \tag{3.19b}$$

and integrate (3.18) by the method of steepest descent. Noting the relation $p(v+2\lambda) = 1/p(v)$, we get

$$-V_+(v)/V_-(v) \sim 1 + \alpha(v)\, p^M(v_s)\, p^{M\eta}\,(v_s - \lambda) + \cdots \tag{3.20}$$

where v_s is the saddle point of $|p(v)\, p^\eta\,(v-\lambda)|$, determined by

$$\eta = -\frac{f(z_s, x^4)\, f(x^2 z_s, x^4)\, f(-x z_s, x^4)\, f(-x^3 z_s, x^4)}{f(-z_s, x^4)\, f(-x^2 z_s, x^4)\, f(x z_s, x^4)\, f(x^3 z_s, x^4)} \tag{3.21a}$$

$$z_s = \exp(-\pi v_s/2I) \tag{3.21b}$$

with the condition

$$v_s = \lambda + 2iI, \qquad \eta = 0 \tag{3.21c}$$

The explicit form of $\alpha(v)$, which is determined by $D(v)$ and the derivative of $p(v)$, is not important here. Since $|p(v_s)\, p^\eta\,(v_s - \lambda)| < 1$, (3.20) shows that the doublet of the largest eigenvalues $V_r(v)$ is asymptotically degenerate as $M \to \infty$.

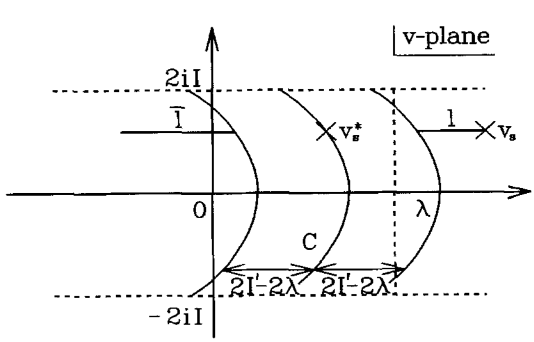

Fig. 7. The two line segments l and $\bar{l}$.

When $I'/2 < \lambda < I'$, there arises a case where the saddle point v_s determined by (3.21) is in the region c. For $v' \in$ the region c, the asymptotic form (3.10c) cannot be used to estimate the logarithms in the integrands of (3.18). Assuming that $v_s \in$ the region c, we define two line segments l and $\bar{l}$ as follows: a point v_s^* on the contour C is chosen so that $\mathrm{Im}(v_s) = \mathrm{Im}(v_s^*)$; l connects the points $v_s^* + 2I' - 2\lambda$ and v_s; $\bar{l}$ connects $v_s - 2\lambda$ and $v_s^* - 2I' + 2\lambda$ (Fig. 7). We calculate $P_+(v) - P_-(v)$ around l and $1/P_+(v) - 1/P_-(v)$ around $\bar{l}$. We use two integral equations:

$$\begin{aligned}
\ln\left[\frac{P_+(v)}{P_-(v)}\right]
&= \ln\left[\frac{1+P_+(v-2I'+2\lambda)}{1+P_-(v-2I'+2\lambda)}\right] + \ln\left[\frac{1+P_+(v)}{1+P_-(v)}\right] \\
&+ \frac{1}{8iI}\int_{v_s-2iI}^{v_s+2iI} dv' \ln\left[\frac{1+P_+(v')}{1+P_-(v')}\right] \\
&\times [D(v-v'-2I') + D(v'-v)] \\
&- \frac{1}{8iI}\int_{v_s-2\lambda-2iI}^{v_s-2\lambda+2iI} dv' \ln\left[\frac{1+1/P_+(v')}{1+1/P_-(v')}\right] \\
&\times [D(v-v'-2I') + D(v'-v)]
\end{aligned} \tag{3.22a}$$

for v around l; and

$$\begin{aligned}
\ln\left[\frac{P_+(v)}{P_-(v)}\right]
&= -\ln\left[\frac{1+1/P_+(v)}{1+1/P_-(v)}\right] - \ln\left[\frac{1+1/P_+(v+2I'-2\lambda)}{1+1/P_-(v+2I'-2\lambda)}\right] \\
&+ \frac{1}{8iI}\int_{v_s-2iI}^{v_s+2iI} dv' \ln\left[\frac{1+P_+(v')}{1+P_-(v')}\right] \\
&\times [D(v-v') + D(v'-v-2I')] \\
&- \frac{1}{8iI}\int_{v_s-2\lambda-2iI}^{v_s-2\lambda+2iI} dv' \ln\left[\frac{1+1/P_+(v')}{1+1/P_-(v')}\right] \\
&\times [D(v-v') + D(v'-v-2I')]
\end{aligned} \tag{3.22b}$$

for v around $\bar{l}$ (Appendix B). Suppose that (3.19a) and (3.19b) hold around l and $\bar{l}$, respectively. Then the logarithms in the integrands of (3.22) are estimated by (3.19); (3.22a) and (3.22b) are integrated by steepest descent.

From the monotonicity of $p(v)\, p^{\eta}(v-\lambda)$ along l (or $\bar{l}$), it follows that the dominant contribution as $M \to \infty$ comes from the first (or second) term on the rhs of (3.22a) [or (3.22b)]. Keeping only the dominant terms, we get

$$\begin{aligned} P_+(v)/P_-(v) &\sim 1 + 2p^M(v - 2I' + 2\lambda) \\ &\qquad \times p^{M\eta}(v - 2I' + \lambda) \qquad \text{for } v \text{ around } l \qquad (3.23\text{a}) \\ &\sim 1 - 2p^M(v + 2I')\, p^{M\eta}(v + 2I' - \lambda) \\ &\qquad\qquad\qquad \text{for } v \text{ around } \bar{l} \qquad (3.23\text{b}) \end{aligned}$$

For v around l (or $\bar{l}$), (3.23a) [or (3.23b)] with (3.10c) derives (3.19a) [or (3.19b)] again. From this fact, though we cannot prove it rigorously, we expect that (3.19a) [or (3.19b)] gives the correct asymptotic form of $P_+(v) - P_-(v)$ [or $1/P_+(v) - 1/P_-(v)$] arouns l (or $\bar{l}$). Therefore, the argument from (3.18) to (3.21) functions even if the saddle point v_s is in the region c.

Now, setting $v = 0$ (or λ), and choosing the values of the ρ_j suitably, we consider the original system (A) [or (B)]. When M and N become large with η fixed to be constant, we can use the asymptotic forms of V_r to estimate the partition functions in (2.4). Substituting (3.12) and (3.20) with $v = 0$ into (2.18) gives

$$\begin{aligned} Z_{MN}^{(n)} &\sim V_+^N(0) + (-)^n\, V_-^N(0) \\ &\sim \begin{cases} 2\kappa^N(0) & \text{for } n \equiv 0 \pmod 2 \\ N\alpha(0)\, p^M(v_s)\, p^{M\eta}(v_s - \lambda)\, \kappa^N(0) & \text{for } n \equiv 1 \pmod 2 \end{cases} \qquad (3.24) \end{aligned}$$

(Note that N is even.) Using (3.24) with (2.3a) in (2.4), we obtain

$$\begin{aligned} \sigma/k_B T = &-\cos\theta_\perp \ln p(v_s - \lambda) \\ &- \sin\theta_\perp \ln p(v_s), \qquad 0 < \theta_\perp < \pi/2 \qquad (3.25) \end{aligned}$$

From the investigation of $v = \lambda$ and the relation $\sigma(-\theta_\perp) = \sigma(\theta_\perp)$, it is found that the expression (3.25) is analytically continued into $-\pi < \theta_\perp < \pi$ with v_s regarded as a function of $\theta_\perp$.

4. EQUILIBRIUM CRYSTAL SHAPE

The ECS of the eight-vertex model is derived from the anisotropic interfacial tension $\sigma(\theta_\perp)$ calculated in Section 3. We represent $\mathbf{R} = (X, Y)$, the position vector of a point on the ECS, as a function of $\theta_\perp$, the slope of the interface at that point. As $\theta_\perp$ varies from $-\pi$ to π, $\mathbf{R}(\theta_\perp)$ sweeps out the ECS. According to Wulff's construction, the ECS is given by

$$\Lambda X = \cos\theta_\perp \sigma - \sin\theta_\perp \, d\sigma/d\theta_\perp \tag{4.1a}$$

$$\Lambda Y = \sin\theta_\perp \sigma + \cos\theta_\perp \, d\sigma/d\theta_\perp \tag{4.1b}$$

where Λ is a scale factor adjusted to yield the volume of the crystal.[7] Substituting (3.25) into (4.1), we find that

$$\Lambda X/k_B T = -\ln p(v_s - \lambda), \qquad \Lambda Y/k_B T = -\ln p(v_s) \tag{4.2}$$

where the saddle point v_s is a function of $\theta_\perp$, determined by (3.21) with (2.3a). As the temperature is lowered, the ECS deforms into a square from a sphere near the critical temperature (Fig. 8).

It is helpful to calculate the radii of curvature for $\theta_\perp = \pi/4$ and $\pi/2$, where $\Lambda\mathbf{R} = \sigma(\cos\theta_\perp, \sin\theta_\perp)$. In the zero-temperature limit, a facet appears at the point $\theta_\perp = \pi/4$; a corner appears at $\theta_\perp = \pi/2$. It follows that

$$\rho/R = 1 + \sigma^{-1} \, d^2\sigma/d\theta_\perp^2, \qquad \theta_\perp = \pi/4, \pi/2 \tag{4.3}$$

where ρ is the radius of curvature and $R = |\mathbf{R}|$.[23] For $\theta_\perp =$

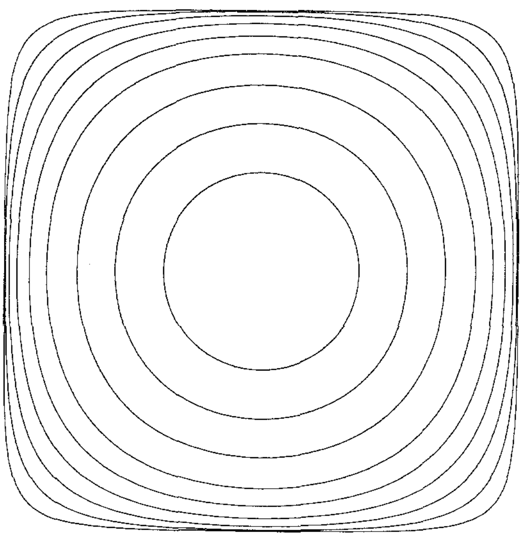

Fig. 8. The equilibrium crystal shape of the eight-vertex model. We rotate the coordinate axes through $\pi/4$, and choose $k_B T \ln[2k^{1/2}(x) + 2k^{-1/2}(x)]$ as the scale factor Λ. From the outermost figure, $x = 1.0 \times 10^{-6}$, 1.0×10^{-4}, 0.001, 0.004, 0.01, 0.02, 0.04, 0.07, and 0.12, successively.

$\pi/4+\delta\theta_\perp$ ($\delta\theta_\perp \approx 0$), using (2.3a) and (3.21), we expand z_s into a power series of $\delta\theta_\perp$ as

$$z_s = -x^{3/2}(1+\Delta^{(1)}\delta\theta_\perp + \Delta^{(2)}\delta\theta_\perp^2 + \cdots) \tag{4.4a}$$

$$\Delta^{(1)} = f^2(x, x^4)\, f^2(-x, x^4)/x^{1/2}Q^3(x^4)\, f(-1, x^4) \times f(x^2, x^4)\, f(-x^2, x^4) \tag{4.4b}$$

$$\vdots$$

where

$$Q(q) = \prod_{n=1}^{\infty} (1-q^n) \tag{4.5}$$

Substituting (4.4) into (3.25) gives

$$\begin{aligned}\frac{\sigma}{k_B T} &= \sqrt{2}\ln\left[x^{-1/4}\frac{f(-x^{3/2}, x^4)}{f(-x^{1/2}, x^4)}\right] \\ &+ \frac{1}{2}\left\{-\sqrt{2}\ln\left[x^{-1/4}\frac{f(-x^{3/2}, x^4)}{f(-x^{1/2}, x^4)}\right]\right. \\ &+ \frac{\sqrt{2}}{x^{1/2}}\frac{f^2(x, x^4)\, f^2(-x, x^4)}{f^2(-1, x^4)\, f^2(-x^2, x^4)} \\ &\left.\times \frac{f(x^{1/2}, x^4)\, f(x^{3/2}, x^4)}{f(-x^{1/2}, x^4)\, f(-x^{3/2}, x^4)}\right\}\delta\theta_\perp^2 + \cdots\end{aligned} \tag{4.6}$$

From (4.3) and (4.6), the radius of curvature at $\theta_\perp = \pi/4$ is calculated as

$$\begin{aligned}\frac{\rho}{R} &= x^{-1/2}\frac{f^2(x, x^4)\, f^2(-x, x^4)}{f^2(-1, x^4)\, f^2(-x^2, x^4)}\frac{f(x^{1/2}, x^4)\, f(x^{3/2}, x^4)}{f(-x^{1/2}, x^4)\, f(-x^{3/2}, x^4)} \\ &\times \left\{\ln\left[x^{-1/4}\frac{f(-x^{3/2}, x^4)}{f(-x^{1/2}, x^4)}\right]\right\}^{-1}\end{aligned} \tag{4.7}$$

In the zero-temperature limit, where $x \to 0$, it follows that

$$\rho/R \sim -1/x^{1/2}\ln x, \qquad \theta_\perp = \pi/4 \tag{4.8}$$

The critical point corresponds to the $I \to \infty$ limit, with λ being of the order of unity. By the use of the conjugate modulus identity (A.6b), we find that near the critical point

$$\rho/R \sim 1 + (16/3)\exp(-2\pi I/\lambda), \qquad \theta_\perp = \pi/4 \tag{4.9}$$

Similarly, for $\theta_\perp = \pi/2 + \delta\theta_\perp$, expanding z_s as

$$z_s = -x(1 + \Delta^{(1)}\delta\theta_\perp + \Delta^{(2)}\delta\theta_\perp^2 + \cdots) \tag{4.10a}$$

$$\Delta^{(1)} = f^2(x, x^4)\, f(-1, x^4)\, f(-x^2, x^4)/Q^3(x^4) \\ \times f^2(-x, x^4)\, f(x^2, x^4) \tag{4.10b}$$

$$\vdots$$

we get

$$\frac{\sigma}{k_B T} = \ln\left[x^{-1/2}\frac{f(-x^2, x^4)}{f(-1, x^4)}\right] \\ + \frac{1}{2}\left\{-\ln\left[x^{-1/2}\frac{f(-x^2, x^4)}{f(-1, x^4)}\right] + \frac{f^4(x, x^4)}{f^4(-x, x^4)}\right\}\delta\theta_\perp^2 \cdots \tag{4.11}$$

The radius of curvature at $\theta_\perp = \pi/2$ is given by

$$\frac{\rho}{R} = \frac{f^4(x, x^4)}{f^4(-x, x^4)} \Big/ \ln\left[x^{-1/2}\frac{f(-x^2, x^4)}{f(-1, x^4)}\right] \tag{4.12}$$

It follows that in the zero-temperature limit

$$\rho/R \sim -2/\ln x, \qquad \theta_\perp = \pi/2 \tag{4.13}$$

and that near the critical temperature

$$\rho/R \sim 1 - (16/3)\exp(-2\pi I/\lambda), \qquad \theta_\perp = \pi/2 \tag{4.14}$$

As mentioned in Section 1, the eight-vertex model contains the six-vertex model and the square-lattice nearest-neighbor Ising model as special limits. In the parametrization (2.1), the eight-vertex model reduces to the six-vertex model as $q \to 0$; the eight-vertex model factors into two independent nearest-neighbor Ising models when $q = x^4$. The expressions of the anisotropic interfacial tension (3.25) and the ECS (4.2) are independent of q, however. It is found that the ECS (4.2) can be rewritten into the compact form

$$\cosh[\Lambda(X + Y)/k_B T] + \cosh[\Lambda(X - Y)/k_B T] \\ = k^{1/2}(x) + k^{-1/2}(x) \tag{4.15}$$

where $k(x)$ is defined by (1.6b). If the coordinate axes are rotated through $\pi/4$ and the scale factor Λ is redefined suitably, (4.15) is identical to the

ECS of the six-vertex model (1.5a) with C_{I} replaced by C_{BC}. For $q = x^4$, it follows from (1.3) and (A.5e) that

$$\sinh(2\,|J_1|\,k_{\mathrm{B}}T) = \sinh(2\,|J_2|/k_{\mathrm{B}}T) = k^{-1/2}(x) \tag{4.16}$$

Using (4.16), we can reproduce the ECS (1.5) from (4.15).

From the beginning of Section 3 until this point we have been investigating the special case $u_0 = 0$. The calculation in Section 3 is easily extended to the general case $-\lambda < u_0 < \lambda$. For $-\lambda < u_0 < \lambda$, the anisotropic interfacial tension is given by

$$\begin{aligned} \sigma/k_{\mathrm{B}}T = & -\cos\theta_\perp \ln p(v_s - u_0 - \lambda) \\ & - \sin\theta_\perp \ln p(v_s), \qquad -\pi < \theta_\perp < \pi \end{aligned} \tag{4.17}$$

where v_s is the saddle point of $|p(v)\, p^\eta(v - u_0 - \lambda)|$, determined by

$$\eta = -\frac{f(z_s a^{-1}, x^4)\, f(x^2 z_s a^{-1}, x^4)\, f(-x z_s, x^4)\, f(-x^3 z_s, x^4)}{f(-z_s a^{-1}, x^4)\, f(-x^2 z_s a^{-1}, x^4)\, f(x z_s, x^4)\, f(x^3 z_s, x^4)} \tag{4.18a}$$

$$z_s = \exp(-\pi v_s/2I), \qquad a = \exp(-\pi u_0/2I) \tag{4.18b}$$

with the condition

$$v_s = \lambda + 2iI, \qquad \eta = 0 \tag{4.18c}$$

and η is related to $\theta_\perp$ by (2.3a). The expressions of the ECS (4.2) and (4.15) are generalized as

$$\Lambda X/k_{\mathrm{B}}T = -\ln p(v_s - u_0 - \lambda), \qquad \Lambda Y/k_{\mathrm{B}}T = -\ln p(v_s) \tag{4.19}$$

and

$$\cosh[\Lambda(X + Y)/k_{\mathrm{B}}T] + A_3 \cosh[\Lambda(X - Y)/k_{\mathrm{B}}T] = -A_4/2 \tag{4.20}$$

respectively. The explicit forms of A_3 and A_4 are given in the following argument.

We pointed out a relation between the form of the elliptic function $p(v)$ and the algebraic curve of the ECS (4.20). We rewrite (4.20) as

$$\alpha^2\beta^2 + 1 + A_3(\alpha^2 + \beta^2) + A_4\alpha\beta = 0 \tag{4.21a}$$

where

$$\alpha = \exp(-\Lambda X/k_{\mathrm{B}}T), \qquad \beta = \exp(-\Lambda Y/k_{\mathrm{B}}T) \tag{4.21b}$$

Equation (4.21a) is a symmetric biquadratic relation between α and β. It is known that this relation is naturally parametrized in terms of Jacobian elliptic functions as

$$\begin{aligned} \alpha &= \hat{k}^{1/2}\,\widehat{\mathrm{sn}}(\zeta+\eta), \qquad & \beta &= \hat{k}^{1/2}\,\widehat{\mathrm{sn}}\,\zeta \\ A_3 &= -1/\hat{k}\,\widehat{\mathrm{sn}}^2\eta, \qquad & A_4 &= 2\,\widehat{\mathrm{cn}}\,\eta\,\widehat{\mathrm{dn}}\,\eta/\hat{k}\,\widehat{\mathrm{sn}}^2\eta \end{aligned} \tag{4.22}$$

(See Chapter 15 of ref. 1.) Here, we define the elliptic modulus $\hat{k}$ by (A.3a) in Appendix A, and the Jacobian elliptic functions $\widehat{\mathrm{sn}}\,u$, $\widehat{\mathrm{cn}}\,u$, and $\widehat{\mathrm{dn}}\,u$ by (A.1) and (A.4), with the elliptic norm $\hat{q}$; the quarter-periods are denoted by $\hat{I}$ and $\hat{I}'$. If we relate the variables in (4.22) to those in (4.19) by

$$\begin{aligned} \hat{I} &= I, \qquad & \hat{I}' &= \lambda \\ \zeta &= i(\lambda - v_s)/2, \qquad & \eta &= i(u_0+\lambda)/2 \end{aligned} \tag{4.23}$$

then (4.22) is identical with (4.19). We stress that the form of $p(v)$ has the source of the symmetric biquadratic relation (4.21a). In this sense, the form of $p(v)$ reflects the algebraic curve of the ECS (4.20).

5. SUMMARY AND DISCUSSION

The anisotropic interfacial tension of the eight-vertex model has been found by the shift operator method. We considered two inhomogeneous systems defined on a square lattice of $(1+\eta)M$ columns and N rows $[(1+\eta)M, N \text{ even}]$. We imposed on the systems periodic boundary conditions along the horizontal direction and antiperiodic boundary conditions along the vertical direction. In each system the lhs of the Mth column was in an antiferroelectric ordered state. An interface ran across this region because of the antiperiodic boundary conditions. The interface was sloped by the rhs of the Mth column, which had the effect of shifting the endpoint of the interface on the Mth column from that on the first column along the vertical direction. The inhomogeneous systems were analyzed by Baxter's commuting transfer matrices argument. It was shown that a doublet of the largest eigenvalues of the row–row transfer matrix is asymptotically degenerate as $M \to \infty$ with η fixed to be constant. The interfacial tension of the sloped interface was calculated from the finite-size correction terms of the doublet of the largest eigenvalues in the $M \to \infty$ limit.

Using the calculated anisotropic interfacial tension, we derived the ECS of the eight-vertex model via Wulff's construction. The eight-vertex model contains the six-vertex model and the square-lattice nearest-neighbor Ising model as the $q \to 0$ and x^4 limits, respectively. The ECSs of these two models have been obtained. It has been shown that they are essentially the same. This fact was extended to the q independence of the ECS of the eight-vertex model. The ECS of the eight-vertex model is a simple algebraic curve in the X–Y plane. We regarded the algebraic curve as a symmetric biquadratic relation between $\alpha = \exp(-\Lambda X/k_B T)$ and $\beta = \exp(-\Lambda Y/k_B T)$. It was shown that an elliptic function appearing in the expression of the interfacial tension has the form to parametrize this symmetric biquadratic relation naturally. In other words, the form of the elliptic function reflects the algebraic curve of the ECS.

Here, we reexamine the ECS of the hard-hexagon model, which was been derived in ref. 19. The anisotropic interfacial tension of the hard-hexagon model is represented as

$$\frac{\sigma}{k_B T} = -\frac{2}{\sqrt{3}}\left[\cos\theta_\perp \ln|\psi(a_s x)| + \cos\left(\theta_\perp - \frac{\pi}{3}\right)\ln|\psi(a_s)|\right],$$
$$-\pi < \theta_\perp < \pi \tag{5.1}$$

where a_s is determined as a function of $\theta_\perp$ by (3.10) of ref. 19; the elliptic function $\psi(a)$ is defined by (3.6b) of ref. 19. By the use of Wulff's construction, the ECS is obtained as

$$\frac{\Lambda X}{k_B T} = -\frac{2}{\sqrt{3}}\ln|\psi(a_s x)| - \frac{1}{\sqrt{3}}\ln|\psi(a_s)|,$$
$$\frac{\Lambda Y}{k_B T} = -\ln|\psi(a_s)| \tag{5.2}$$

We can rewrite (5.2) in the compact form

$$\alpha^2\beta^2 + A_1\alpha\beta + (\alpha + \beta) = 0 \tag{5.3a}$$

where

$$\alpha = \exp[-\sqrt{3}\,\Lambda(X + \sqrt{3}\,Y)/2k_B T], \qquad \beta = \exp(\sqrt{3}\,\Lambda X/k_B T) \tag{5.3b}$$

The hard-hexagon model is regarded as a special case of the hard-square model.[1] We can easily extend the analysis in Section 3 of ref. 19 into regime II of the hard-square model. For regime II of the hard-square model, (5.3a) is generalized as

$$B\alpha^3\beta^3 + A_1\alpha^2\beta^2 + \alpha\beta(\alpha + \beta) + A_4\alpha\beta + A_6 = 0 \tag{5.3a$'$}$$

We do not know how (5.3a′) is parametrized in terms of elliptic functions. In Chapter 15 of ref. 1, however, it is shown that the symmetric biquadratic relations between α and β

$$A_1\alpha^2\beta^2+A_2\alpha\beta(\alpha+\beta)+A_3(\alpha^2+\beta^2)+A_4\alpha\beta$$
$$+A_5(\alpha+\beta)+A_6=0 \tag{5.4}$$

are naturally parametrized in terms of elliptic theta functions, the form being

$$\alpha=\phi(u+\eta), \qquad \beta=\phi(u) \tag{5.5a}$$

where

$$\phi(u)=\xi H(u+\gamma)\,H(u-\gamma)/H(u+\delta)\,H(u-\delta) \tag{5.5b}$$

and $H(u)$ is the elliptic theta function defined by (A.1a) in Appendix A. Equation (5.3a) is a special case of (5.4). We relate the norm q and the argument u of the theta functions to x and a_s, the variables in (5.1) and (5.2), by

$$q^2=x^3, \qquad \exp(-i\pi u/I)=a_s \tag{5.6}$$

Set $\xi=x^{1/3}$, $\gamma=\eta=-2iI'/3$, and $\delta=0$ in (5.5b). Then, the elliptic solution (5.2) is reproduced from (5.5a). Thus, the form of $\psi(a)$ appearing in (5.1) is directly related to the algebraic curve of the ECS (5.3a) [or (5.3a′)].

Besides the hard-hexagon model and the eight-vertex model, there are some models whose interfacial tension has been calculated along a special direction. For the Sogo–Akutsu–Abe model[24] and the regimes III and IV of the hard-square model,[25] the interfacial tension is written in terms of the same elliptic function that appears in the expression of the interfacial tension of the eight-vertex model. We expect that the ECSs of these two models are represented as an algebraic curve identical to the ECS of the eight-vertex model. This will be clarified in a further investigation.

APPENDIX A

We define the elliptic theta functions with the norm q and the argument u by

$$H(u)=2q^{1/4}\sin\frac{\pi u}{2I}\prod_{n=1}^{\infty}\left(1-2q^{2n}\cos\frac{\pi u}{I}+q^{4n}\right)(1-q^{2n}) \tag{A.1a}$$

$$H_1(u)=2q^{1/4}\cos\frac{\pi u}{2I}\prod_{n=1}^{\infty}\left(1+2q^{2n}\cos\frac{\pi u}{I}+q^{4n}\right)(1-q^{2n}) \tag{A.1b}$$

$$\Theta(u) = \prod_{n=1}^{\infty} \left(1 - 2q^{2n-1} \cos \frac{\pi u}{I} + q^{4n-2}\right)(1 - q^{2n}) \tag{A.1c}$$

$$\Theta_1(u) = \prod_{n=1}^{\infty} \left(1 + 2q^{2n-1} \cos \frac{\pi u}{I} + q^{4n-2}\right)(1 - q^{2n}) \tag{A.1d}$$

The quarter-periods are given by

$$I = \frac{\pi}{2} \prod_{n=1}^{\infty} \left(\frac{1 + q^{2n-1}}{1 - q^{2n-1}} \frac{1 - q^{2n}}{1 + q^{2n}}\right)^2 \tag{A.2a}$$

$$I' = -\pi^{-1} I \ln q \tag{A.2b}$$

The modulus k and the conjugate modulus k' are

$$k = 4q^{1/2} \prod_{n=1}^{\infty} \left(\frac{1 + q^{2n}}{1 + q^{2n-1}}\right)^4 \tag{A.3a}$$

$$k' = \prod_{n=1}^{\infty} \left(\frac{1 - q^{2n-1}}{1 + q^{2n-1}}\right)^4 \tag{A.3b}$$

By the use of the theta functions, Jacobian elliptic functions are represented as

$$\mathrm{sn}\, u = k^{-1/2} H(u)/\Theta(u) \tag{A.4a}$$

$$\mathrm{cn}\, u = (k'/k)^{1/2}\, H_1(u)/\Theta(u) \tag{A.4b}$$

$$\mathrm{dn}\, u = k'^{1/2} \Theta_1(u)/\Theta(u) \tag{A.4c}$$

With q replaced by $\bar{q} = q^{1/2}$, $\bar{H}(u)$, $\bar{\Theta}(u)$, and $\bar{k}$ are given by (A.1a), (A.1c), and (A.3a), respectively:

$$\bar{H}(u) = 2\bar{q}^{1/4} \sin \frac{\pi u}{2I} \prod_{n=1}^{\infty} \left(1 - 2\bar{q}^{2n} \cos \frac{\pi u}{I} + \bar{q}^{4n}\right)(1 - \bar{q}^{2n}) \tag{A.5a}$$

$$\bar{\Theta}(u) = \prod_{n=1}^{\infty} \left(1 - 2\bar{q}^{2n-1} \cos \frac{\pi u}{I} + \bar{q}^{4n-2}\right)(1 - \bar{q}^{2n}) \tag{A.5b}$$

$$\bar{k} = 4\bar{q}^{1/2} \prod_{n=1}^{\infty} \left(\frac{1 + \bar{q}^{2n}}{1 + \bar{q}^{2n-1}}\right)^4 \tag{A.5c}$$

We define the $\overline{\mathrm{sn}}$ function by

$$\overline{\mathrm{sn}}\, u = \bar{k}^{-1/2} \bar{H}(u)/\bar{\Theta}(u) \tag{A.5d}$$

Then, $\mathrm{sn}(u)$ and $\overline{\mathrm{sn}}(u)$ satisfy the relation

$$\bar{k}\,\overline{\mathrm{sn}}\,u = 2k^{1/2}\,\mathrm{sn}\,u/(1+k\,\mathrm{sn}^2 u) \tag{A.5e}$$

Equation (A.5e) is essentially Landen's transformation.

We define $f(p, q)$ as

$$f(p,q) = (1-p)\prod_{n=0}^{\infty}(1-pq^n)(1-p^{-1}q^n)(1-q^n) \tag{A.6a}$$

The function $f(p, q)$ satisfies the conjugate modulus identity

$$\begin{aligned} ie^{-iv}f(e^{2iv}, e^{-\varepsilon}) = {} & \left(\frac{2\pi}{\varepsilon}\right)^{1/2}\exp\left(\frac{\varepsilon}{8}-\frac{\pi^2}{2\varepsilon}+\frac{2v(\pi-v)}{\varepsilon}\right) \\ & \times f\left(\exp-\frac{4\pi v}{\varepsilon}, \exp-\frac{4\pi^2}{\varepsilon}\right) \end{aligned} \tag{A.6b}$$

APPENDIX B

From the conditions (i)–(iii) given in Section 3 we derive Eqs. (3.10), (3.12), (3.16), and (3.22).[1,18] We start by defining four functions $X_{\pm}(v)$ and $Y_{\pm}(v)$ by

$$\ln X_+(v) = \frac{1}{4iI}\int_{\lambda+\beta-2iI}^{\lambda+\beta+2iI}\frac{\ln[1+P_r(v')]}{\exp[\pi(v-v')/2I]-1}\,dv', \qquad \mathrm{Re}(v) > \lambda+\beta \tag{B.1a}$$

$$\ln X_-(v) = \frac{-1}{4iI}\int_{\lambda+\beta'-2iI}^{\lambda+\beta'+2iI}\frac{\ln[1+P_r(v')]}{\exp[\pi(v-v')/2I]-1}\,dv', \qquad \mathrm{Re}(v) < \lambda+\beta' \tag{B.1b}$$

$$\ln Y_+(v) = \frac{1}{4iI}\int_{-\beta'-2iI}^{-\beta'+2iI}\frac{\ln[1+1/P_r(v')]}{\exp[\pi(v-v')/2I]-1}\,dv', \qquad \mathrm{Re}(v) > -\beta' \tag{B.1c}$$

$$\ln Y_-(v) = \frac{-1}{4iI}\int_{-\beta-2iI}^{-\beta+2iI}\frac{\ln[1+1/P_r(v')]}{\exp[\pi(v-v')/2I]-1}\,dv', \qquad \mathrm{Re}(v) < -\beta \tag{B.1d}$$

where $0 < \beta < \beta' < \delta$. From Cauchy's residue theorem, it follows that

$$X_+(v)\,X_-(v) = 1+P_r(v), \qquad \lambda+\beta < \mathrm{Re}(v) < \lambda+\beta' \tag{B.2a}$$

$$Y_+(v)\,Y_-(v) = 1+1/P_r(v), \qquad -\beta' < \mathrm{Re}(v) < -\beta \tag{B.2b}$$

Using (B.2a), we define $X_+(v)$ for $\mathrm{Re}(v) \leqslant \lambda + \beta$ and $X_-(v)$ for $\mathrm{Re}(v) \geqslant \lambda + \beta'$. The condition (ii) shows that $X_+(v)$ is ANZ if v is on the rhs of the contour C, and that $X_-(v)$ is ANZ if $\mathrm{Re}(v) < \lambda + \delta$. Similarly, (B.2b) is used to define $Y_+(v)$ for $\mathrm{Re}(v) \leqslant -\beta'$ and $Y_-(v)$ for $\mathrm{Re}(v) \geqslant -\beta$. It is found that $Y_+(v)$ is ANZ if $\mathrm{Re}(v) > -\delta$, and $Y_-(v)$ if v is on the lhs of C. Substituting (B.2) into (3.2), we find that

$$V_r(v) = \phi_2(v)\, q(v - 2\lambda')\, X_+(v)\, X_-(v)/q(v) \tag{B.3a}$$

$$V_r(v) = \phi_1(v)\, q(v + 2\lambda')\, Y_+(v)\, Y_-(v)/q(v) \tag{B.3b}$$

The functions $\phi_1(v)$, $\phi_2(v)$, and $q(v)$ are rewritten as

$$\phi_1(v) = \rho_1^M \rho_{M+1}^{M\eta} \gamma^{2m} z^m x^{-m} \exp\left(\frac{m\pi}{2I} \frac{\eta}{1+\eta} \lambda\right)$$

$$\times A(\lambda - v)\, A^\eta(2\lambda - v)\, A(2I' - \lambda + v)\, A^\eta(2I' - 2\lambda + v) \tag{B.4a}$$

$$\phi_2(v) = \rho_1^M \rho_{M+1}^{M\eta} \gamma^{2m} z^{-m} x^{-m} \exp\left(-\frac{m\pi}{2I} \frac{\eta}{1+\eta} \lambda\right)$$

$$\times A(\lambda + v)\, A^\eta(v)\, A(2I' - \lambda - v)\, A^\eta(2I' - \lambda) \tag{B.4b}$$

$$q(v) = \gamma^m \exp[\pi(mv - v_1 - v_2 - \cdots - v_m)/4I]\, F(v)\, G(v - 2I') \tag{B.4c}$$

$$= (-\gamma)^m \exp[\pi(v_1 + v_2 + \cdots + v_m - mv)/4I]\, F(v + 2I')\, G(v) \tag{B.4d}$$

where γ and $A(v)$ are given by (3.14) and (3.15), respectively; $F(v)$ and $G(v)$ are defined as

$$F(v) = \prod_{j=1}^{m} \prod_{n=0}^{\infty} \{1 - q^n \exp[-\pi(v - v_j)/2I]\} \tag{B.5a}$$

$$G(v) = \prod_{j=1}^{m} \prod_{n=0}^{\infty} \{1 - q^n \exp[\pi(v - v_j)/2I]\} \tag{B.5b}$$

Substitute (B.4) into (B.3); use (B.4c) for $q(v)$ in (B.3a) and $q(v + 2\lambda')$ in (B.3b); use (B.4d) for $q(v - 2\lambda')$ in (B.3a) and $q(v)$ in (B.3b). Then, it follows that

$$V_r(v) = r\rho_1^M \rho_{M+1}^{M\eta} \gamma^{2m} x^{-2m} L_+(v)\, L_-(v) \tag{B.6a}$$

$$= r\rho_1^M \rho_{M+1}^{M\eta} \gamma^{2m} x^{-2m} M_+(v)\, M_-(v) \tag{B.6b}$$

where

$$L_+(v) = A(\lambda + v)\, A^\eta(v)\, F(v + 2I' - 2\lambda)\, X_+(v)/F(v) \tag{B.7a}$$

$$L_-(v) = A(2I' - \lambda - v)\, A^\eta(2I' - v)\, G(v - 2\lambda)\, X_-(v)/G(v - 2I') \tag{B.7b}$$

$$M_+(v) = A(2I' - \lambda + v)\, A^\eta(2I' - 2\lambda + v) \times F(v + 2\lambda)\, Y_+(v)/F(v + 2I') \tag{B.7c}$$

$$M_-(v) = A(\lambda - v)\, A^\eta(2\lambda - v)\, G(v + 2\lambda - 2I')\, Y_-(v)/G(v) \tag{B.7d}$$

Noting (iii), we find that $L_+(v)$ is ANZ for $\mathrm{Re}(v) > 0$, $L_-(v)$ for $\mathrm{Re}(v) < \delta'$, $M_+(v)$ for $\mathrm{Re}(v) > \lambda - \delta'$, and $M_-(v)$ for $\mathrm{Re}(v) < \lambda$, with $\delta' = \min\{2I' - \lambda, 2\lambda, \lambda + \delta\}$. From (B.6), we get

$$M_+(v)/L_+(v) = L_-(v)/M_-(v) \tag{B.8}$$

Both sides of (B.8) are entire, since the lhs is ANZ for $\mathrm{Re}(v) > 0$ and the rhs is ANZ for $\mathrm{Re}(v) < \lambda$. Moreover, they are bounded in the $\mathrm{Re}(v) \to \pm\infty$ limit. From Liouville's theorem, it follows that they are constant. The fact that the rhs of (B.8) $\to 1$ as $\mathrm{Re}(v) \to \infty$ shows that this constant is 1. Finally, we obtain

$$M_+(v) = L_+(v) \tag{B.9a}$$

$$M_-(v) = L_-(v) \tag{B.9b}$$

Two functions $S_\pm(v)$ are introduced as

$$S_+(v) = F(v + 2\lambda)\, F(v)/A(v + \lambda)\, A^\eta(v) \tag{B.10a}$$

$$S_-(v) = G(v)\, G(v - 2\lambda)/A(\lambda - v)\, A^\eta(2\lambda - v) \tag{B.10b}$$

Equation (B.9a) gives the recursion relations for $S_+(v)$

$$S_+(v) = S_+(v + 2I' - 2\lambda)\, X_+(v)/Y_+(v) \tag{B.11a}$$

Similarly, from (B.9b), it follows that

$$S_-(v) = S_-(v - 2I' + 2\lambda)\, Y_-(v)/X_-(v) \tag{B.11b}$$

Equations (B.11) are solved as

$$S_+(v) = \prod_{n=0}^{\infty} X_+[v + 2n(I' - \lambda)]/Y_+[v + 2n(I' - \lambda)] \tag{B.12a}$$

$$S_-(v) = \prod_{n=0}^{\infty} Y_-[v - 2n(I' - \lambda)]/X_-[v - 2n(I' - \lambda)] \tag{B.12b}$$

Regarding (B.10a) and (B.10b) as a recursion relations for $F(v)$ and $G(v)$, respectively, we find that

$$F(v)=\prod_{n=0}^{\infty}\frac{A[v+(4n+1)\lambda]\,A^{\eta}(v+4n\lambda)\,S_{+}(v+4n\lambda)}{A[v+(4n+3)\lambda]\,A^{\eta}[v+(4n+2)\lambda]\,S_{+}[v+(4n+2)\lambda]} \tag{B.13a}$$

$$G(v)=\prod_{n=0}^{\infty}\frac{A[(4n+1)\lambda-v]\,A^{\eta}[(4n+2)\lambda-v]\,S_{-}(v-4n\lambda)}{A[(4n+3)\lambda-v]\,A^{\eta}[(4n+4)\lambda-v]\,S_{-}[v-(4n+2)\lambda]} \tag{B.13b}$$

Substitute (B.4) into (3.1); (B.4c) is used for $q(v+2\lambda')$; (B.4d) is used for $q(v-2\lambda')$. Then, using (B.13), we obtain the expression

$$\begin{aligned}P_r(v)&=rp^{M}(v)\,p^{M\eta}(v-\lambda)\\&\times\prod_{n=0}^{\infty}\frac{S_{+}[v+(4n+2)\lambda]\,S_{+}(v+2I'+4n\lambda)}{S_{+}[v+(4n+4)\lambda]\,S_{+}[v+2I'+(4n-2)\lambda]}\\&\times\prod_{n=0}^{\infty}\frac{S_{-}[v-2I'-(4n-2)\lambda]\,S_{-}[v-(4n+4)\lambda]}{S_{-}(v-2I'-4n\lambda)\,S_{-}[v-(4n+2)\lambda]}\end{aligned} \tag{B.14a}$$

where $p(v)$ is defined by (3.11). By the use of the periodicity relation (2.17b), Eq. (3.1) is rewritten as

$$P_r(v)=rsq^{-m/2}x^{-2m-M\eta}z^{m}\phi_1(v)\,q(v+2\lambda')/\phi_2(v)\,q(v-2\lambda'-2I') \tag{3.1$'$}$$

$$=rsq^{m/2}x^{-2m+M\eta}\phi_1(v)\,q(v+2\lambda'-2I')/\phi_2(v)\,q(v-2\lambda') \tag{3.1$''$}$$

Replacing (3.1) by (3.1′) or (3.1″) in the derivation of (B.14a), we get

$$P_r(v)=S_{+}(v)\,S_{-}(v-2I'+2\lambda)/S_{+}(v-2\lambda)\,S_{-}(v-2I') \tag{B.14b}$$

$$\begin{aligned}&=p^{M}(v)\,p^{M\eta}(v-\lambda)\,p^{M}(v-2I')\,p^{M\eta}(v-\lambda-2I')\\&\times\prod_{n=0}^{\infty}\frac{S_{+}[v-2I'+(4n+2)\lambda]\,S_{+}(v+2I'+4n\lambda)}{S_{+}[v-2I'+(4n+4)\lambda]\,S_{+}[v+2I'+(4n-2)\lambda]}\\&\times\prod_{n=0}^{\infty}\frac{S_{-}[v-4I'-(4n-2)\lambda]\,S_{-}[v-(4n+4)\lambda]}{S_{-}(v-4I'-4n\lambda)\,S_{-}[v-(4n+2)\lambda]}\end{aligned} \tag{B.14c}$$

Substituting (B.12) into (B.14c) gives the integral equations (3.22). From (B.12a), it follows that $S_{+}(v)$ is exponentially close to 1 as $M\to\infty$ if v is on the rhs of C. Equation (B.12b) shows that $S_{-}(v)$ is exponentially close to 1 when M is large and v is on the lhs of C. For $v\in$ the region a and M

large, $S_+(v)$ and $S_-(v)$ on the rhs of (B.14a) can be replaced by 1. So, we get (3.10a) as $M \to \infty$. Similarly, for $v \in$ the regions b and c, (3.10b) and (3.10c) are found from (B.14b) and (B.14c), respectively. [The three regions a, b, and c are defined in the argument following (3.11).]

Replace $L_-(v)$ by $M_-(v)$ in (B.6a); substitute (B.7a) and (B.7d) into (B.6a); and use (B.13). Then, it follows that

$$V_r(v) = r\kappa(v) \left\{ X_+(v) \prod_{n=0}^{\infty} \frac{S_+[v+2I'+(4n-2)\lambda]\, S_+[v+(4n+2)\lambda]}{S_+(v+2I'+4n\lambda)\, S_+(v+4n\lambda)} \right.$$
$$\left. \times Y_-(v) \prod_{n=0}^{\infty} \frac{S_-[v-2I'-(4n-2)\lambda]\, S_-[v-(4n+2)\lambda]}{S_-(v-2I'-4n\lambda)\, S_-(v-4n\lambda)} \right\} \quad \text{(B.15)}$$

where $\kappa(v)$ is defined by (3.13). Using (B.12) in (B.15), we get the integral equation (3.16). On the rhs of (B.15) the contribution from the curly bracket is exponentially close to 1 for $0 < \mathrm{Re}(v) < \lambda$ and large M. Thus, we obtain (3.12).

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